Robust solutions for the Kidney Exchange Problem

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1 Problem definition

The *Kidney Exchange Problem* (KEP) entails finding a *best* set of kidney exchanges in a pool of altruistic donors and incompatible patient-donor pairs. Once a KEP solution is obtained, it is proposed to the practitioners who perform more accurate medical exams which may reveal that some transplants need to be cancelled due a wrong evaluation of the medical compatibility between some patients and donors. Hence, we study robust solution for the KEP in the settings of [2].

The KEP can be defined on a directed weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ referred to as *compatibility graph*. The vertex set is $\mathcal{V} = \mathcal{I} \cup \mathcal{D}$ where $\mathcal{I}$ and $\mathcal{D}$ are the sets of incompatible patient-donor pairs and altruistic donors, respectively. $\mathcal{A}$ contains an arc (i, j) if $i, j \in \mathcal{V}$ are compatible for a transplant. We assign a weight W_{ij} to each $(i, j) \in \mathcal{A}$ representing the medical benefit of the transplant. Kidney exchanges are modelled as circuits and paths in $\mathcal{G}$ of length at most L^C and L^P. Their set is $\mathcal{E}$. The KEP aims to determine a union of pairwise vertex-disjoint exchanges of maximum weight. We study the *Robust KEP under Existence Uncertainty* (RKEP-EU) where the existence of the arcs in the compatibility graph is subject to uncertainty. The RKEP-EU determines a best solution for the KEP when at most Γ arcs are allowed to fail. Under the worst-case scenario, removing one arc from a KEP solution entails removing the entire circuit/path containing it. Thus, the uncertainty set for the RKEP-EU can be expressed in terms of exchange weights: $\mathcal{U}_\Gamma^E = \left\{\tilde{\mathbf{W}} = (\tilde{W}^e)_{e\in\mathcal{E}} : \tilde{W}^e = W^e(1-\delta_e), \delta_e \in \{0,1\}, \sum_{e\in\mathcal{E}} \delta_e \leq \Gamma\right\}$. The aim of the RKEP-EU is to determine a KEP solution whose worst-case realization of the uncertain exchange weights within uncertainty set $\mathcal{U}_\Gamma^E$ is maximal.

2 Problem formulation and solution method

The max-min formulation for the RKEP-EU reads as follows: [max-min^E] $\max_{\boldsymbol{\lambda}\in\Lambda} \min_{\tilde{\mathbf{W}}\in\mathcal{U}_\Gamma^E} \sum_{e\in\mathcal{E}} \tilde{W}^e \lambda_e$, where $\Lambda = \{\boldsymbol{\lambda} = (\lambda)_{e\in\mathcal{E}} \in \{0,1\}^{|\mathcal{E}|} : \sum_{e\in\mathcal{E}} a_i^e \lambda_e \leq 1 \, \forall i \in \mathcal{V}\}$ is the feasible region of the *cycle formulation* for the KEP. Note that the λ variables are exponentially-many.

A reformulation of [max-min^E] as a maximisation problem is obtained via [1]. Such problem corresponds to a *formulation with column-dependent-rows* which are complex to solve [3]. We

TAB. 1: Results on the instances of the RKEP-EU.

Instances			BPC			[2]		
$\|\mathcal{I}\|$	Γ	#	#opt.	avg.t[s]	avg.gap[%]	#opt.	avg.t[s]	avg.gap[%]
50	1	15	15	5.00	0.00	14	358.35	14.92
	2	15	15	48.30	0.00	12	253.61	26.17
	3	15	15	17.78	0.00	11	337.84	54.86
	4	15	14	21.12	0.86	10	364.76	83.74
	5	15	15	289.29	0.00	9	721.27	140.62
250	1	15	6	589.12	0.70	0	TL	45.52
	2	15	5	165.57	2.31	0	TL	50.46
	3	15	5	204.53	3.40	0	TL	55.73
	4	15	5	190.53	4.17	0	TL	59.37
	5	15	5	272.44	5.23	0	TL	67.08

introduce a more tractable representative-based reformulation of such a problem:

$$[\text{r-S-R}^E(\underline{\mu},\bar{\mu})] \quad \max_{\boldsymbol{\lambda}\in\Lambda,\, \boldsymbol{\varrho}\geq\mathbf{0},\, \underline{\mu}\leq\mu\leq\bar{\mu}} \sum_{e\in\mathcal{E}} W^e\lambda_e - \sum_{h\in\mathcal{V}} \varrho_h - \Gamma\mu \tag{1}$$

$$\text{s.t.} \quad \sum_{e\in\mathcal{E}_h} (W^e-\underline{\mu})^+\lambda_e - \varrho_h - \mu \leq -\underline{\mu} \qquad \forall h\in\mathcal{V} \tag{2}$$

$$\sum_{e\in\mathcal{E}_h} (W^e-\bar{\mu})^+\lambda_e - \varrho_h \leq 0 \qquad \forall h\in\mathcal{V}, \tag{3}$$

where $(a)^+ = \max\{a, 0\}$. $[\text{r-S-R}^E(\underline{\mu},\bar{\mu})]$ is a maximisation problem with exponentially-many variables λ. We propose a Branch-Price-and-Cut (BPC) algorithm to solve it. Precisely, we adapt the one of [4] for the KEP to handle the $(\cdot)^+$ operator appearing in the constraints.

3 Computational experiments

We generated instances for the RKEP-EU by generating KEP instances via a well-know generator from the literature with $|\mathcal{I}| \in \{50, 250\}$, $|\mathcal{D}| = 10\%|\mathcal{I}|$, graph density of 10%, $L^C = 3$ and $L^P = 3, 6, 12$. We assign values to budget $\Gamma \in \{1, 2, 3, 4, 5\}$.

The results from Table 1 reveal that our approach consistently outperforms the state-of-the-art approach for the RKEP-EU in [2] in terms of number of optima (*#opt.*), average time to obtain them (*avg.t[s]*) and average optimality gap on the instances not solved to optimality (*avg.gap[%]*).

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