FIRST AND SECOND ORDER OPTIMALITY CONDITIONS FOR NONSMOOTH MULTIOBJECTIVE PROBLEMS WITH EQUILIBRIUM CONSTRAINTS

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Abstract

In this work, we first consider a multiobjective optimization problem with mixed constraints, indicated by MOP, as follows:

$$\min \quad \mathfrak{f}(\mathfrak{y}) := (\mathfrak{f}_1(\mathfrak{y}), \ldots, \mathfrak{f}_p(\mathfrak{y})) \quad \text{subject to} \quad \mathfrak{y} \in \mathcal{S}, \tag{MOP}$$

where

$$\mathcal{S} := \Big\{ \mathfrak{y} \in \mathbb{R}^r : \quad \mathfrak{g}_i(\mathfrak{y}) \leq 0, \quad \forall i \in \mathcal{M} := \{1, \ldots, m\}, \\ \mathfrak{h}_j(\mathfrak{y}) = 0, \quad \forall j \in \mathcal{N} := \{1, \ldots, n\} \Big\}$$

is the feasible region of the problem (MOP) and $\mathfrak{f}_t : \mathbb{R}^r \to \mathbb{R}, t \in \mathcal{P} := \{1, \ldots, p\}$ $\mathfrak{g}_i : \mathbb{R}^r \to \mathbb{R}, i \in \mathcal{M}$ and $\mathfrak{h}_j : \mathbb{R}^r \to \mathbb{R}, j \in \mathcal{N}$ are locally Lipschitz functions. We introduce a constant positive linear dependence (CPLD) condition in terms of convexificators for the MOP, denoted by MOP-CPLD, as follows:

Definition 1 ***(CPLD condition for MOP)*** *Let* $\varkappa \in \mathcal{S}$, *let* $\mathfrak{f}_t, t \in \mathcal{P}$ *and* $\mathfrak{g}_i, i \in \mathcal{M}$ *admit upper semi-regular convexificator (USRC)* $\partial^* \mathfrak{f}_t(\mathfrak{y}), t \in \mathcal{P}$ *and* $\partial^* \mathfrak{g}_i(\mathfrak{y}), i \in \mathcal{M}$, *respectively, and let* $\mathfrak{h}_j, j \in \mathcal{N}$ *admits a pseudo-differential (PD)* $\partial \mathfrak{h}_j(\mathfrak{y}), j \in \mathcal{N}$ *at each* $\mathfrak{y}$ *in a nbd of* $\varkappa$. *For each* $\mathcal{P}_1^s \subseteq \mathcal{P}_s, s \in \mathcal{P}$, $\mathcal{M}_1 \subseteq \mathcal{M}(\varkappa)$ *and* $\mathcal{N}_1 \subseteq \mathcal{N}$, *consider* $\hat{\varkappa}_{\mathfrak{f}_t} \in conv\partial^* \mathfrak{f}_t(\varkappa), t \in \mathcal{P}_1^s, s \in \mathcal{P}$, $\hat{\varkappa}_{\mathfrak{g}_i} \in conv\partial^* \mathfrak{g}_i(\varkappa), i \in \mathcal{M}_1$, *and* $\hat{\varkappa}_{\mathfrak{h}_j} \in conv\partial \mathfrak{h}_j(\varkappa), j \in \mathcal{N}_1$. *Then, the CPLD condition for MOP, indicated by MOP-CPLD, is satisfied at* $\varkappa$ *iff for each sequence* $\{\varkappa_\alpha\}_{\alpha \in \mathbb{N}} \subseteq \mathbb{R}^r \to \varkappa$ *with* α *sufficiently large, the set of subgradients* $\{\hat{\varkappa}^\alpha_{\mathfrak{f}_t} : t \in \mathcal{P}_1^s\} \cup \{\hat{\varkappa}^\alpha_{\mathfrak{g}_i} : i \in \mathcal{M}_1\} \cup \{\hat{\varkappa}^\alpha_{\mathfrak{h}_j} : j \in \mathcal{N}_1\}$ *is linearly dependent whenever for any* $s \in \mathcal{P}$, *the set* $\{\hat{\varkappa}_{\mathfrak{f}_t} : t \in \mathcal{P}_1^s\} \cup \{\hat{\varkappa}_{\mathfrak{g}_i} : i \in \mathcal{M}_1\} \cup \{\hat{\varkappa}_{\mathfrak{h}_j} : j \in \mathcal{N}_1\}$ *is positively linearly dependent. Here the subgradients is defined as* $\hat{\varkappa}^\alpha_{\mathfrak{f}_t} \in conv\partial^* \mathfrak{f}_t(\varkappa_\alpha), \hat{\varkappa}^\alpha_{\mathfrak{g}_i} \in conv\partial^* \mathfrak{g}_i(\varkappa_\alpha), \hat{\varkappa}^\alpha_{\mathfrak{h}_j} \in conv\partial \mathfrak{h}_j(\varkappa_\alpha)$ *for* $t \in \mathcal{P}_1^s, i \in \mathcal{M}_1, j \in \mathcal{N}_1$, *and they satisfy the convergence conditions* $\hat{\varkappa}^\alpha_{\mathfrak{f}_t} \to \hat{\varkappa}_{\mathfrak{f}_t}, \hat{\varkappa}^\alpha_{\mathfrak{g}_i} \to \hat{\varkappa}_{\mathfrak{g}_i}$ *and* $\hat{\varkappa}^\alpha_{\mathfrak{h}_j} \to \hat{\varkappa}_{\mathfrak{h}_j}$.

The MOP-CPLD condition extends the CPLD condition in terms of convexificators given by Rimpi and Lalitha [1] for nonsmooth scalar optimization problems and the CPLD condition given by Andreani et al. [2] involving continuously differentiable functions. We establish a strong Karush-Kuhn-Tucker (KKT) optimality condition to identify local Pareto efficient solutions under the MOP-CPLD framework as follows:

Theorem 1 *Let $\varkappa \in \mathcal{S}^{le}_{MOP}$ such that the the MOP-CPLD condition is satisfied at $\varkappa$. Then, there exist $\mu^{\mathfrak{f}}_t > 0, t \in \mathcal{P}, \mu^{\mathfrak{g}}_i \geq 0, i \in \mathcal{M}$ and $\mu^{\mathfrak{h}}_j \in \mathbb{R}, j \in \mathcal{N}$, such that*

$$0 \in \sum_{t \in \mathcal{P}} \mu^{\mathfrak{f}}_t conv\partial^* \mathfrak{f}_t(\varkappa) + \sum_{i \in \mathcal{M}} \mu^{\mathfrak{g}}_i conv\partial^* \mathfrak{g}_i(\varkappa) + \sum_{j \in \mathcal{N}} \mu^{\mathfrak{h}}_j conv\partial \mathfrak{h}_j(\varkappa), \tag{1a}$$

$$\mu^{\mathfrak{g}}_i \mathfrak{g}_i(\varkappa) = 0, \quad \forall i \in \mathcal{M}. \tag{1b}$$

Further, we study a multiobjective problem with equilibrium constraints, indicated by MOPEC, as follows:

$$\min \quad \mathfrak{f}(\mathfrak{y}) := (\mathfrak{f}_1(\mathfrak{y}), \ldots, \mathfrak{f}_p(\mathfrak{y})) \quad \text{subject to} \quad \mathfrak{y} \in \mathcal{F} \tag{MOPEC}$$

where

$$\begin{aligned} \mathcal{F} := \{\mathfrak{y} \in \mathbb{R}^r : \quad & \mathfrak{g}_i(\mathfrak{y}) \leq 0, \quad \forall i \in \mathcal{M}, \\ & \mathfrak{h}_j(\mathfrak{y}) = 0, \quad \forall j \in \mathcal{N}, \\ & \mathfrak{G}_k(\mathfrak{y}) \geq 0, \mathfrak{H}_k(\mathfrak{y}) \geq 0, \quad \forall k \in \mathcal{L} := \{1, \ldots, l\}, \\ & \mathfrak{G}_k(\mathfrak{y}) \mathfrak{H}_k(\mathfrak{y}) = 0, \quad \forall k \in \mathcal{L}\} \end{aligned}$$

and $\mathfrak{f}_t, t \in \mathcal{P}, \mathfrak{g}_i, i \in \mathcal{M}, \mathfrak{h}_j, j \in \mathcal{N}, \mathfrak{G}_k, k \in \mathcal{L}, \mathfrak{H}_k, k \in \mathcal{L}$ are real-valued functions defined on $\mathbb{R}^r$. We also introduce a suitable CPLD condition for the MOPEC, denoted by MOPEC-CPLD. We introduce several nonsmooth strong Pareto stationary points for the MOPEC which extend the notions of strong Pareto stationary points given by Zhang et al. [3] for continuously differentiable functions. We provide necessary and sufficient optimality conditions to identify a stationary point as a Pareto efficient solution of the MOPEC under the MOPEC-CPLD condition.

Moreover, we introduce second order Abadie constraint qualifications for MOPEC which is denoted by MOPEC-SOACQ in terms of Clarke generalized derivative and second-order upper directional derivative given by Páles and Zeidan [4]. This notion utilizes second-order ACQ given by Anchal and Lalita [5] for multiobjective optimization problems. We derive second-order necessary optimality conditions in both the primal and the dual forms to identify weak Pareto efficient solutions and strict Pareto efficient solutions of order two for MOPEC by utilizing MOPEC-SOACQ. We give some applications of the results in interval-valued multiobjective optimization problems with equilibrium constraints and in portfolio optimization.

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